Introductio

in Elementarem Functionum Ellipticarum Theoriam,

quam

Cons. Ampliss. Philos. Ord. Lundensi,

Præside

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Pro Laurea

miti bonorum censuræ

offert

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$Z_0 = \int \frac{du}{\sqrt{2^1(u^2)}}$, $Z_1 = \int \frac{u^2 du}{\sqrt{2^1(u^2)}}$, et $\Pi_1 = \int \frac{du}{(u^2 - a)\sqrt{2^1(u^2)}}$, ad quas jam p. 4. alia via pervenimus. Hæ vero, vel ipsis similes F, E et Π p. 23 occurrentes, *tria functionum Ellipticarum genera* constituunt, (*Nomicam*, *Epinomicam* et *Paranomicam* dictas). Vice versa vero, si proponeretur forma Πn pag. 3 occurrens, faceremus $1 + ay^2 = x$, $2aydy = dx$, $2^0 y^2 = 2^1 x$, et $(x-1) . 2^0 x = 3^0 x$, foretque $\Pi n = \int \frac{dx}{2ax^n \sqrt{\frac{x-1}{a} . 2^1 x}} =$

$\frac{1}{2\sqrt{a}} . \int \frac{dx}{x^n \sqrt{3^0 x}}$, quam ex modo annotatis facile ad formas Z_0, Z_1 et Π_1 reducas. Similiter, proposita ejusdam loci forma Z_n, facias $y^2 = x$, moxque tibi exorietur $Z_n = \int \frac{y^{2n} dy}{\sqrt{2^0(y^2)}} = \int \frac{x^n . dx}{2\sqrt{x . 2^0 x}} = \int \frac{x^n dx}{\sqrt{3^0 x}}$, quam modo ad $X_0 = \int \frac{dx}{\sqrt{3^0 x}}$ et $X_1 = \int \frac{x dx}{\sqrt{3^0 x}} = \int dx . \sqrt{\frac{x}{2^0 x}}$ ideoque ad Z_0 et Z_1 revocavimus. Sin vero viam pag. 3. indicatam præoptaris, facile invenies, (si $2^0 y^2 = 2_0 + 2_1 y^2 + 2_2 y^4$), $A_n = (2n-1) . 2_2$, $B_n = (2n-2) . 2_1$, $C_n = (2n-3) . 2_0$.

Sin vero ad formas Π_n ascendere velis, invenies A_n, B_n, C_n et D_n ibidem occurrentes resolvendo hanc (ex $d\left(\frac{y\sqrt{2^0 y^2}}{(1 + ay^2)^{n-1}}\right)$ ortam) fractionem

$\frac{(2^0(y^2) + 2_1 y^2 + 2 . 2_2 y^4).(1 + ay^2) + 2a.\overline{1-n}.y^2 . 2^0 y^2}{(1 + ay^2)^n}$ in partiales

$\frac{A_n}{\overline{1 + ay^2}^n} + \frac{B_n}{\overline{1 + ay^2}^{n-1}} + \frac{C_n}{\overline{1 + ay^2}^{n-2}} + \frac{D_n}{\overline{1 + ay^2}^{n-3}}$; id quod optime, faciendo $1 + ay^2 = z$, perficitur.

Sic ex. gr. mox invenitur $A_n = 2a.\overline{1-n}.\left(\frac{-1}{a}\right).2^0\left(\frac{-1}{a}\right) =$ $2.\overline{n-1}.2^0\left(\frac{-1}{a}\right)$, unde edocemur, n haud ita accipi posse, ut Π_1 per Π_0, Π_{-1} et Π_{-2} exprimatur, nisi forte $2^0\left(\frac{-1}{a}\right) = o$ fuerit, seu $1+ay^2$ factor ipsius $2^0(y^2)$, quo casu (facto $n=2$) $B_2\Pi_1 + C_2\Pi_0 + D_2\Pi_{-1} = \frac{y\sqrt{2^0y^2}}{1+ay^2}$, ideoque Π_1 per Π_0 et Π_{-1} habetur, B_2 enim haud evanescit, nisi $\sqrt{2^0y^2}$ rationalis fuerit.

Facto scilicet $2^0(y^2) = (1+ay^2)(b+y^2)$, fit $d\left(y\sqrt{\frac{b+y^2}{1+ay^2}}\right) =$

$$\sqrt{\frac{b+y^2}{1+ay^2}} + \frac{(1-ab)y^2}{(1+ay^2)\sqrt{2^0y^2}} = \frac{1}{a}.\left(\sqrt{\frac{1+ay^2}{b+y^2}} + \frac{ab-1}{(1+ay^2)\sqrt{2^0y^2}}\right),$$

unde $(ab-1).\int\frac{dy}{(1+ay^2)\sqrt{2^0y^2}} = -\int dy\sqrt{\frac{1+ay^2}{b+y^2}} + ay\sqrt{\frac{b+y^2}{1+ay^2}}$; nec fit $ab-1=o$, nisi $2^0y = (1+ay^2)\left(\frac{1}{a}+y^2\right) = \frac{(1+ay^2)^2}{a}$, quare hoc integrale semper ita reducitur.

Ad pag. 2 et 54.

Proposita vero formula $\frac{Ydy}{\sqrt{4^1.y}}$ integranda, ubi $4^1y = y^4 + 2ay^3 + by^2 + 2cy + e$, prævie secundus tollatur terminus faciendo $y = x - \frac{1}{2}a$, ideoque

$$4^1y = x^4 + \left(b - \frac{3}{2}a^2\right)\left(x^2 - ax + \frac{a^2}{4}\right) + \left(2c - \frac{a^3}{2}\right)\left(x - \frac{a}{2}\right) + e - \left(\frac{a}{2}\right)^4,$$

quæ in duos factores deinde resolvatur. Scilicet, si jam factum fu-

erit $a=o$, ut sit $4^1y=4^0x=x^4+bx^2+2cx+e$, hæc functio ponatur $=(x^2+2lx+m).(x^2-2lx+n)$, qui factores nominentur $=2^0x.2^1x$, unde hæ æquationes exoriuntur $b=m+n-2^2l^2$, $c=l.\overline{n-m}$, $e=mn$, quæ statim ad æquationem $4e=(\overline{m+n}^2-\overline{n-m}^2=)(4l^2+b)^2-\frac{c^2}{l^2}$, cubicam in l^2, conducunt. Cujus igitur radice l^2 inventa, ideoque n et m $\left(=2l^2+\frac{b}{2}\pm\frac{1}{2}\cdot\frac{c}{l}\right)$ datis, faciatur $x=\frac{p+qu}{1+u}$, ponaturque $(1+u)^2.2^0x$ vel $(1+u)^2.2^1x$ inde formam $P+Qu^2$ nancisci; quem in finem ipsius u coefficientes $pq+l(p+q)+m$ et $pq-l\overline{p+q}+n$ necessario nihilo $(=o)$ æquentur, ideoque $p+q=\frac{n-m}{2.l}=\frac{c}{2l^2}$ et $pq=\frac{n+m}{2}=-\frac{b}{2}-2l^2$ faciantur. Inde vero p et $q\left(=\frac{c}{4l^2}\pm\sqrt{\frac{c^2}{4^2l^4}+2l^2+\frac{b}{2}}\right)$ quadratice definiuntur, et quidem realiter, si radicem l^2 rite elegeris, (ut sit $\left(\frac{c}{4l^2}\right)^2+2l^2>-\frac{b}{2}\Big)$. Id quod semper succedit; si enim ipsius $4^0x=o$ radices sunt x_0, x_1, x_2, x_3, fit $2l=x_0+x_1=-x_2-x_3$, $n=x_0x_1$ et $m=x_2.x_3$, ideoque $(p-q)^2=(p+q)^2-4pq=\left(\frac{x_0x_1-x_2x_3}{x_0+x_1}\right)^2+$ $+2.\overline{x_0x_1+x_2x_3}$, id quod, evolutione complete instituta, hacque reductione $x_0+x_1+x_2+x_3=o$ adhibita, huic producto $\frac{(x_0-x_2)(x_0-x_3)(x_1-x_2)(x_1-x_3)}{(x_0+x_1)^2}$ convenire tibi facile persuadeas. Hic vero numerator positivus erit, si x_0 et x_1 ipsas x_2 et x_3 superaverint, vel etiam utræque utrisque inferiores fuerint, vel si $x_0>x_2>x_3>x_1$. Sin vero x_2 et x_3 imaginariæ fuerint et radices æquationis $2^0x=o$, necessario erunt producta $\overline{x_0-x_2}.\overline{x_0-x_3}=2^0x_0$ et $\overline{x_1-x_2}.\overline{x_1-x_3}=2^0x_1$

positiva, ideoque et $(p-q)^2$ positivum, atque p et q realia. Idem et valebit, si etiam x_0 et x_1 imaginariæ fuerint, tum enim $2^0 x_0$ et $2^0 x_1$ formæ erunt $X_0 \pm i X_1$, ideoque $2^0 x_0 . 2^0 x_1 = X_0^2 + X_1^2 > o$. Præterea $p-q$ haud evanescit, nisi radicum $x_0\, x_1\, x_2$ et x_3 duæ æquales fuerint, ex. gr. $x_2 = x_3$, quo casu $4^0 x = \overline{x - x_2}^2 . 2^1 x$ et $\int \frac{dx}{\sqrt{4^0 x}} = \int \frac{dx}{\overline{x - x_2} \sqrt{2^1 x}}$ haud elliptica. — Realiter igitur inventis p et q, eisque diversis, erit $dx = \frac{(q-p)du}{(1+u)^2}$ et $(1+u)^2 . 2^0 x = 2^0 p + 2^0 q . u^2 = (p-q).(p + l - \overline{q + l} u^2)$ similiterque $(1+u)^2 . 2^1 x = 2^1 p + 2^1 q . u^2 = (p-q).(p - l + \overline{l - q} u^2)$. Facto enim $2^0 x = x^2 + 2lx + m$, $x = \frac{p+qu}{1+u}$, invenitur $2^0 p = p^2 + 2lp + m = p^2 + lp - (pq + lq) = (p+l)(p-q)$ et $2^0 q = q^2 + 2lq + m = (q+l)(q-p)$, indeque $2^1 p$ et $2^1 q$, mutando l in $-l$, et m in n, (ut sit $2^1 x = x^2 - 2lx + n$), obtinentur.

Erit igitur $\frac{dx}{\sqrt{4^0 x}} = \frac{(q-p).du}{\sqrt{(1+u)^2 . (2^0 x) . (1+u)^2 . 2^1 x}} =$

$$\pm \frac{du}{\sqrt{(p + l - \overline{q + l} u^2)(p - l + \overline{l - q} u^2)}} = \frac{du}{\sqrt{2^0 (u^2)}},$$ si $2^0 (u^2) = (p + l - \overline{q + l} u^2)(p - l + \overline{l - q} u^2) = \overline{p - q u^2}^2 - l^2 \overline{1 - u^2}^2)$, ideoque formæ hypercyclicæ. Simul vero factor rationalis Y vel X in similem U ipsius u functionem migrabit, deindeque $\int \frac{U du}{\sqrt{2^0 (u^2)}}$ ex regulis modo traditis ad formas Z_0, Z_1 et Π_1 revocabis. Sic ex. gr. speciatim erit

$$\pm \frac{(x^2 + \alpha)dx}{\sqrt{4^0 x}} = \frac{(\overline{p + qu}^2 + \alpha \overline{1 + u}^2)du}{\overline{1+u}^2 \sqrt{(p + l - \overline{q + l} u^2)(p - l + \overline{l - q} u^2)}} = dV,$$ si $4^0 x = x^4 + bx^2 + 2cx + e$. Jam ad hanc formam ulterius reducendam faciamus

obiter $1+u=v$, eritque $\frac{\overline{p+qu}^2+\alpha\,\overline{1+u}^2}{\overline{1+u}^2} = \alpha + \frac{\overline{p-q+qv}^2}{v^2} =$

$$= \alpha + q^2 + 2.\frac{\overline{p-q}.q}{v} + \frac{\overline{p-q}^2}{v^2}\text{, ideoque } dV =$$

$$= \frac{du}{\sqrt{2^0u^2}}.\left(\alpha+q^2+2.\frac{\overline{p-q}.q}{1+u}+\frac{\overline{p-q}^2}{\overline{1+u}^2}\right)\text{. Est vero } d\left(\frac{\sqrt{2^0u^2}}{1+u}\right) =$$

$$= \frac{du}{\sqrt{2^0u^2}}.\left(\frac{2u(\overline{1-u^2}\,l^2-q\,\overline{p-qu^2})}{1+u} - \frac{\overline{p-qu^2}^2-l^2\,\overline{1-u^2}^2}{\overline{1+u}^2}\right)$$

seu, substitutione & evolutione rite instituta, $d\left(\frac{\sqrt{2^0u^2}}{1+u}\right) =$

$$= \frac{du}{\sqrt{2^0(u^2)}}.\left(\overline{q^2-l^2}.\overline{v^2-2v} + 2q.\frac{\overline{q-p}}{v} - \frac{\overline{p-q}}{v^2}\right)\text{, ideoque,}$$

(quia $v^2-2v=u^2-1$, &c.) hæc addendo et integrando,

$$V + \frac{\sqrt{2^0u^2}}{1+u} = (\alpha+l^2)\int\frac{du}{\sqrt{2^0u^2}} + (q^2-l^2)\int\frac{u^2du}{\sqrt{2^0u^2}}.$$

Integrale igitur $V=\int\frac{(x^2+\alpha)dx}{\sqrt{x^4+bx^2+2cx+e}}$ ad formam $\int\frac{(\alpha^1+\beta^1u^2)du}{\sqrt{2^0(u^2)}}$ seu ad functiones Ell. *primi secundique* generis dictas revocatur.

Ex. 2. Sit deinde $U=\int\frac{(x+\alpha)dx}{\sqrt{4^0x}}$, eritque, ut vidimus, $\frac{dx}{\sqrt{4^0x}}=\frac{du}{\sqrt{2^0(u^2)}}$,

ideoque $dU = \frac{(p+qu+\alpha\,\overline{1+u})}{\overline{1+u}.\sqrt{2^0(u^2)}} = \frac{(\alpha(1-u^2)+p+\overline{q-p}\,u-qu^2)\,du}{(1-u^2)\sqrt{2^0(u^2)}}$, seu

$d\,U = \frac{du}{\sqrt{2^0(u^2)}} \cdot \left(\alpha + q + \frac{p-q}{1-u^2}\right) + \frac{(q-p)\,.\,u\,du}{(1-u^2)\sqrt{2^0(u^2)}}$, unde, facto $u^2 = w$, $\int \frac{(x+\alpha)dx}{\sqrt{4^0 x}} = \frac{(q-p)}{2} \int \frac{dw}{\overline{1-w}\sqrt{2^0 w}} + (\alpha+q) \int \frac{du}{\sqrt{2^0(u^2)}} +$

$+(p-q)\int \frac{du}{\overline{1-u^2}\sqrt{(2^0 u^2)}}$, quare hæc forma ad functionem Ell. *primi* & *tertii* generis revocatur, quarum illa tum deest, cum $\alpha = -q$, hæc vero semper adest, quandoquidem $p-q$ haud evanescit, nisi $\sqrt{4^0 x}$ in $\overline{x-x_0} \cdot \sqrt{2^0 x}$ sponte redigitur. Sin vero U aliter, et quidem logarithmice, ut interdum licet, invenitur, tum certe hæc tertii generis $\int \frac{du}{\overline{1-u^2}\sqrt{2^0 u^2}}$ in *primam* mutabitur. Cujus quidem rei mox exemplum videbis.

Ex. 3. Sit demum $W = \int \frac{dx}{\overline{x+\alpha}\sqrt{4^0 x}}$; erit, ut antea, $\frac{dx}{\sqrt{4^0 x}} = \frac{du}{\sqrt{2^0(u^2)}}$, ideoque si $p + \alpha = P$, $q + \alpha = -Q$, $W = \int \frac{(1+u)\,du}{(P-Qu)\sqrt{2^0(u^2)}}$, et igitur, si fuerit $\alpha = -q$, seu $Q = o$, $u^2 = w$, $W = \frac{1}{P} \int \frac{du}{\sqrt{2^0 u^2}} +$ $+ \frac{1}{2.P} \int \frac{dw}{\sqrt{2^0 w}}$, seu hoc casu W *primi* fit generis. Alias vero erit :

$$W = \int \frac{(1+u)\,(P+Qu)\,du}{(P^2-Q^2u^2)\sqrt{2^0 u^2}} = \int \frac{(P+Qu^2)\,du}{(P^2-Q^2u^2)\sqrt{2^0 u^2}} + \frac{P+Q}{2} \int \frac{dw}{(P^2-Q^2 w)\sqrt{2^0 w}},$$

seu $W = (P+Q) \cdot \left(\frac{P}{Q} \int \frac{du}{(P^2-Q^2u^2)\sqrt{2^0 u^2}} + \frac{1}{2} \int \frac{dw}{\overline{P^2-Q^2 w}\sqrt{2^0 w}}\right) - \frac{1}{Q} \int \frac{du}{\sqrt{2^0 u^2}}$,

nec igitur functio tertii generis (Π) deerit, nisi fuerit α vel $= -q$ vel $= -p$, nam $P+Q=p-q$ haud evanescit. — Sit ex. gr. $4^0x = x^4+2cx^2+1 = 2^0x.2^1x$, et $c^2<1$, eritque $2^0x = x^2 \pm 2lx+1$, si $l=\sqrt{\frac{1-c}{2}}$, $m=1=n$, ideoque $p+q=\frac{n-m}{2}=o$, $q=-p$, et $-p^2=pq=-\frac{n+m}{2}=-1$, unde $p=1$, $q=-1$, ideoque faciendum $x=\frac{1-u}{1+u}$; fietque ita

$$\frac{dx}{\sqrt{4^0x}}=\frac{-du}{\sqrt{(1+l+\overline{1-l}u^2)(1-l+\overline{1+l}u^2)}}=-\frac{du}{\sqrt{2^0u^2}}$$ atque 1)

$$(-V=)\int\frac{(x^2+\alpha)dx}{\sqrt{x^4+2cx^2+1}}=\frac{\sqrt{2^0u^2}}{1+u}-(\alpha+l^2)\int\frac{du}{\sqrt{2^0u^2}}+(l^2-1)\int\frac{u^2du}{\sqrt{2^0u^2}}.$$

Si igitur $c<1$ fuerit, erit l cum formula $\frac{du}{\sqrt{2^0u^2}}$ reale, ut p. **54.** Geometrice ostendimus. 2) Simul vero hoc casu obtinetur

$$(U=)\int\frac{(x+\alpha)dx}{\sqrt{4^0x}}=-\alpha\int\frac{d(u^2)}{(1-u^2)\sqrt{2^0u^2}}-\int\frac{2du}{\overline{1-u^2}\sqrt{2^0u^2}}+\int\frac{du}{\sqrt{2^0(u^2)}}.$$

3) Atque $(W(1-\alpha)=)\int\frac{(1-\alpha)dx}{(x+\alpha)\sqrt{4^0x}}=$

$$=\int\frac{2.\overline{1+\alpha}du}{(\overline{1+\alpha}^2-\overline{1-\alpha}^2u^2)\sqrt{2^0u^2}}-\int\frac{du}{\sqrt{2^0u^2}}-\int\frac{d\widehat{u^2}}{1^0\widehat{u^2}.\sqrt{2^0u^2}}.$$

Præterea hoc universim monendum, si aliquando formula obtenta $\frac{du}{\sqrt{2^0(u^2}}\left(\text{pro }\frac{dx}{\sqrt{4^0x}}\right)$ eis haud prædita fuerit signis, quæ maxime exoptares, substitutionem formæ $u^2=\frac{\alpha+\beta y^2}{\gamma+\delta y^2}$ ex pag. 20 petendam adhiberi,

vel etiam hanc (p. 54 adhibitam) $u = \zeta . \frac{\lambda + y}{\lambda - y}$ in usum vocari. Scil. si inventum fuerit $2^0(u^2) = \pm (A + u^2)(B + u^2)$, exsistentibus A & B positivis vel negativis, numeros α, β, γ & δ ita eligere licet, ut fiat $\frac{du}{\sqrt{2^0 u^2}} = \frac{dy}{\sqrt{2' y^2}}$ atque $2' y^2$ ex-gr. hujus præoptatæ formæ $C(1 - y^2)(1 - c^2 y^2)$. Vel etiam λ & ζ ita eligere licet, ut similis forma exoriatur? Fit enim $2'(y^2) =$

$$= \pm (A(\lambda - y)^2 + \zeta^2 \overline{\lambda + y}^2).(B \overline{\lambda - y}^2 + \zeta^2 \overline{\lambda + y}^2) =$$

$$= \pm (\overline{A + \zeta^2}\ \overline{\lambda^2 + y^2} + 2\lambda \overline{\zeta^2 - A}\, y).(\overline{B + \zeta^2}.\overline{\lambda^2 + y^2} + 2\lambda \overline{\zeta^2 - B}\, y),$$

ideoque $2'(y^2) = \pm \overline{A + \zeta^2}.\overline{B + \zeta^2}.(\overline{\lambda^2 + y^2}^2 - C^2 y^2)$, siquidem $C = 2\lambda . \frac{\zeta^2 - A}{A + \zeta^2} = -2\lambda . \frac{\zeta^2 - B}{\zeta^2 + B}$ fuerit, ideoque $\zeta^4 = AB$. Hæc igitur substitutio in usum vocari potest, quoties A & B eisdem prædita sunt signis.

Quin imo, ut ad formam generalem $\frac{dx}{\sqrt{4^0 x}}$ adhuc paululum redeamus, exsistente $4^0 x = C \overline{x - \alpha}\ \overline{x - \beta}\ \overline{x - \gamma}\ \overline{x - \delta}$ functione quarti gradus quacunque $(= (x^4 + ax^3 + bx^2 + cx + e)C)$, ipsam et aliter et quidem simplicissime, (quoties hujus radices duæ saltem reales fuerint) ad formam $dz : \sqrt{2^0(z^2)}$ revocare licet, faciendo $x - \alpha = \overline{\delta - x}\, z^2$, seu $x = \frac{\alpha + \delta z^2}{1 + z^2}$. Ita enim fit

$$\frac{dx}{\sqrt{4^0 x}} = \frac{(\delta - \alpha).2z\,dz}{\sqrt{-C.\overline{x - \delta}^2.z^2.\overline{1 + z^2}^2.(\alpha + \delta z^2 - \beta \overline{1 + z^2})(\alpha + \delta z^2 - \gamma \overline{1 + z^2})}} =$$

$$= \frac{2\,dz}{\sqrt{C(\alpha - \beta + \overline{\delta - \beta}\, z^2)(\gamma - \alpha + \overline{\gamma - \delta}\, z^2)}}.$$ Sin vero $x - \alpha : \delta - x$ negativa foret, $-u^2$ pro z^2 adhiberes, vel etiam radices α, β, γ, δ permutares.